Axiomatizations for universal classes

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Example 1: Birkhoff

Identities are FO sentences of the form

$$\forall \bar{x} \ \varphi(\bar{x}),$$

where φ is atomic: $\varphi = t(\bar{x}) \approx s(\bar{x})$ or $\varphi = R(t_1(\bar{x}), \ldots, t_n(\bar{x}))$.

HS(**A**) - the class of homomorphic images of substructures of **A**
P($\mathcal{C}$) - the class of direct products of structures from $\mathcal{C}$.

Birkhoff
A class of structures (in a fixed FO language) is HS and P- closed iff it is definable by identities.

Example 1: main trick

A - a structure

x_a - a fresh variable for every element a from the carrier A.

$$\chi_{\mathbf{A}}^{\mathrm{HS}} = \exists \bar{x} \bigwedge_{\substack{\varphi \text{ atomic} \\ \mathbf{A} \not\models \varphi(a_1,\ldots,a_n)}} \neg\, \varphi(x_{a_1}, \ldots, x_{a_n}).$$

Fact

$\mathbf{B} \models \chi_{\mathbf{A}}^{\mathrm{HS}}$ iff $\mathbf{A} \in \mathrm{HS}(\mathbf{B})$.

Corollary

A class is HS-closed iff it is definable by universal positive (possibly infinite) sentences.

Proof: If $\mathcal{C}$ is a HS-closed class, then it is definable by $\{\neg\chi_{\mathbf{A}}^{\mathrm{HS}} \mid \mathbf{A} \notin \mathcal{C}\}$.

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Now Birkhoff follows by the closure under direct products.

($\mathbf{A} \not\models R(a)$ and $\mathbf{B} \not\models S(b)$ yield $\mathbf{A} \times \mathbf{B} \not\models R(a) \vee S(b)$)

Example 2: Łoś-Tarski and Mal'cev

Quasi-identities are FO sentences of the form

$$\forall \bar{x} \;\; \psi_1(\bar{x}) \wedge \cdots \wedge \psi_k(\bar{x}) \rightarrow \varphi(\bar{x})$$

where φ and all ψs are atomic.
S(**A**) - the class of isomorphic images of substructures of **A**
$\mathrm{P_U}(\mathcal{C})$ - the class of ultraproducts of structures from $\mathcal{C}$.

Mal'cev
A class is S, P and $\mathrm{P_U}$-closed iff it is definable by quasi-identities.

Łoś-Tarski
A class is S and $\mathrm{P_U}$-closed iff it is definable by FO universal sentences.

Fact

A class is S-closed iff it is definable by universal (possibly infinite) sentences.

Proof: Let

$$\chi_{\mathbf{A}}^{\mathsf{S}} = \exists \bar{x} \bigwedge_{\substack{\varphi \text{ atomic} \\ \mathbf{A} \not\models \varphi(a_1,\ldots,a_n)}} \neg\, \varphi(x_{a_1}, \ldots, x_{a_n}) \quad \wedge \bigwedge_{\substack{\psi \text{ atomic} \\ \mathbf{A} \models \psi(a_1,\ldots,a_n)}} \psi(x_{a_1}, \ldots, x_{a_n})$$

Then $\mathbf{B} \models \chi_{\mathbf{A}}^{\mathsf{S}}$ iff $\mathbf{A} \in \mathsf{S}(\mathbf{B})$
and an S-closed class $\mathcal{C}$ is definable by $\{\neg\chi_{\mathbf{A}}^{\mathsf{S}} \mid \mathbf{A} \notin \mathcal{C}\}$.

Fact

A class is S-closed iff it is definable by universal (possibly infinite) sentences.

Proof: Let

$$\chi^{\mathsf{S}}_{\mathbf{A}} = \exists \bar{x} \bigwedge_{\substack{\varphi \text{ atomic} \\ \mathbf{A} \not\models \varphi(a_1,\dots,a_n)}} \neg\, \varphi(x_{a_1}, \dots, x_{a_n}) \quad \wedge \bigwedge_{\substack{\psi \text{ atomic} \\ \mathbf{A} \models \psi(a_1,\dots,a_n)}} \psi(x_{a_1}, \dots, x_{a_n})$$

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Compactness

Let $\mathcal{C}$ be P_{U}-closed. If $\mathcal{C} \models \forall \bar{x} \ \bigvee_{i \in I} \neg\psi_i(\bar{x}) \ \vee \ \bigvee_{j \in J} \varphi_j(\bar{x})$,
then there are finite $I' \subseteq I$ and $J' \subseteq J$ such that
$\mathcal{C} \models \forall \bar{x} \ \bigvee_{i \in I'} \neg\psi_i(\bar{x}) \ \vee \ \bigvee_{j \in J'} \varphi_j(\bar{x})$.

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Now Łoś-Tarski and Mal'cev follow.

Main trick generally

A class operator O is unary iff for every $\mathcal{C}$

$$\mathrm{O}(\mathcal{C}) = \bigcup\{\mathrm{O}(\mathbf{A}) \mid \mathbf{A} \in \mathcal{C}\}.$$

With every structure $\mathbf{A}$ (in a fixed language) associate a sentence $\chi_{\mathbf{A}}$, say in $L_{\infty,\infty}$.
Sentences $\chi_{\mathbf{A}}$ are characteristic for O iff for every $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{B} \models \chi_{\mathbf{A}} \quad \text{iff} \quad \mathbf{A} \in \mathrm{O}(\mathbf{B}).$$

Main Observation

If $\chi_{\mathbf{A}}$ are characteristic sentences for a unary class operator O, then every O-closed class $\mathcal{C}$ is definable by $\{\neg\chi_{\mathbf{A}} \mid \mathbf{A} \notin \mathcal{C}\}$.

Strategy for axiomatization

Find characteristic sentences for O which are also preserved by O.

Example 3: strong homomorphisms

A homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ is strong if $R^{\mathbf{B}} = h(R^{\mathbf{A}})$ for every relational symbol R different than $\approx$. It means that

$$\mathbf{B} \models R(\bar{b}) \quad \text{iff} \quad \exists\, \bar{a} \in A^n \;\; h(\bar{a}) = \bar{b} \text{ and } \mathbf{A} \models R(\bar{a}).$$

$\mathsf{H}_{\mathsf{sng}}\mathsf{S}(\mathbf{A})$ - the class of strong homomorphic images of substructures of $\mathbf{A}$

A quasi-identity is weak if its premise has no occurrences of $\approx$ and functional symbols and every variable appears at most once in the premise.

Theorem

A class $\mathcal{C}$ is $\mathsf{H}_{\mathsf{sng}}\mathsf{S}$ and P-closed iff it is definable by weak quasi-identities.

Closure under the operator	Restrictions on defining sentences	
	in the negative part no occurrences of	in the positive part no occurrences of
S		functional symbols
$\mathsf{H_E S}$	relational symbols	functional symbols
$\mathsf{H_{Str} S}$	$\approx$	
$\mathsf{H_{uSng} S}$	$\approx$, functional symbols	
$\mathsf{H_{Sng} S}$	$\approx$, functional symbols, repetitions of variables	
HS	whole part	
$\mathsf{H_E^{-1} S}$		relational symbols, functional symbols
$\mathsf{H_{Str}^{-1} S}$		$\approx$, functional symbols
$\mathsf{H^{-1} S}$		whole part
$\mathsf{H_E^{-1} H_E S}$	relational symbols	relational symbols
$\mathsf{H_E^{-1} H_{Str} S}$	$\approx$	relational symbols
$\mathsf{H_E^{-1} H_{uSng} S}$	$\approx$, functional symbols	relational symbols
$\mathsf{H_E^{-1} HS}$	whole part	relational symbols
$\mathsf{H_E H_{Str}^{-1} S}$	relational symbols	$\approx$, functional symbols
$\mathsf{H_{Str} H_{Str}^{-1} S}$	$\approx$	$\approx$
$\mathsf{H_{uSng} H_{Str}^{-1} S}$	$\approx$, functional symbols	$\approx$
$\mathsf{H H_{Str}^{-1} S}$	whole part	$\approx$

The end

This is all Thank you!